Math 1510 Week 13

Power Series

A power series is an expression of the form

$$p(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where the constant a is called the center

Rmk $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n = \lim_{m\to 0} \underbrace{\sum_{n=0}^{m} c_n(x-a)^n}_{\text{Polynomial of degree} \le m}$

eg $\displaystyle f(x) = \sum_{n=0}^{\infty} x^n$

$$= 1 + x + x^2 + x^3 + \cdots$$

$f(\frac{1}{2}) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$ (convergent)

$f(2) = 1 + 2 + 4 + 8 + \cdots = \infty$

$f(-1) = 1 - 1 + 1 - 1 + \cdots$ ← divergent

Q For what $x \in \mathbb{R}$ does the series

$$p(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ converges}$$

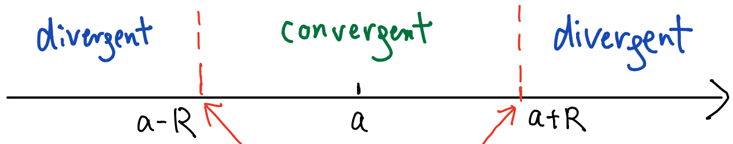
Thm Let $\displaystyle p(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

Then $\exists R$, $R \in \mathbb{R}$ or $R = +\infty$ such that

$$p(x) \begin{cases} \text{converges if } |x-a| < R \\ \text{diverges if } |x-a| > R \end{cases}$$

R is called the Radius of convergence of p(x).

Rmk If $R \in \mathbb{R}$, then p(x) is :



p(x) may or may not converge at $x = a \pm R$.

If $R = +\infty$, then p(x) converges for any $x \in \mathbb{R}$

Q How to find R ?

Thm Let $p(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

If $\lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$ exists or $= +\infty$, then

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right|$$

eg. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n} = (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$

$c_n = \frac{1}{n}$ for $n \geq 1$

$$\lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$$

$\therefore R = 1$

eg $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

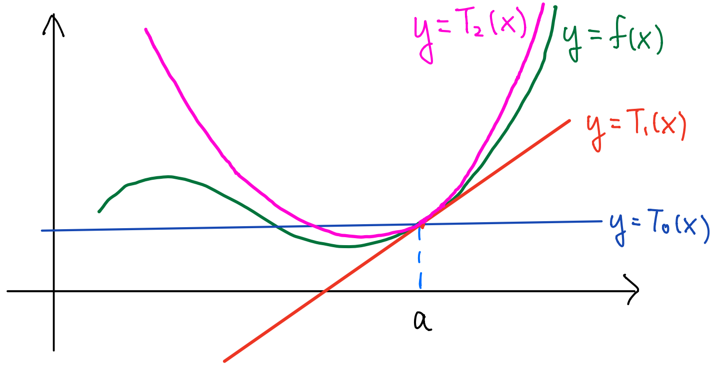
$c_n = \frac{1}{n!}$ for $n \geq 0$

$$\lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| = \lim_{n\to\infty} (n+1) = \infty$$

$\therefore R = \infty$

Taylor Polynomial and Taylor Series

Goal: Approximate $f(x)$ near a by a polynomial of deg $\leq n$



"Best" approximation :

$n=0$: $T_0(x) = f(a)$ $\quad$ T_0, f have same value at a

$n=1$: $T_1(x) = f(a) + f'(a)(x-a)$ $\quad$ T_1, f have same value and slope at a

$n=2$: $T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$

T_2, f have same value, slope, concavity at a

i.e. $T_2(a) = f(a)$ $\quad T_2'(a) = f'(a)$ $\quad T_2''(a) = f''(a)$

Defn Let $f(x)$ be a function, $a \in \mathbb{R}$. Define

① The **n-th order Taylor polynomial** of f at a is defined to be

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

② The **Taylor series** of $f(x)$ at a is defined to be

$$T_\infty(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

③ The **Maclaurin polynomial (series)** of $f(x)$ is defined to be the Taylor polynomial (series) of $f(x)$ at $a=0$

Rmk
- $f^{(0)}(a) = f(a)$
- $0! = 1$
- $T_n^{(k)}(a) = T_\infty^{(k)}(a) = f^{(k)}(a)$ for $0 \le k \le n$

eg Find the Taylor Series of $f(x) = \ln x$ at 1.

Sol $f'(x) = \frac{1}{x}$ $\quad f''(x) = \frac{-1}{x^2}$

$f^{(3)}(x) = \frac{(-1)(-2)}{x^3}$ $\quad f^{(4)}(x) = \frac{(-1)(-2)(-3)}{x^4}$

Similarly, for $k \geq 1$

$$f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{x^k}$$

$$f^{(k)}(1) = (-1)^{k+1}(k-1)!$$

$$\therefore T_\infty(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

$$= \sum_{k=1}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k \quad (\because f(1)=0)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k-1)!}{k!}(x-1)^k$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k$$

eg Let $f(x) = \cos x$

① Find the Maclaurin series

② Approximate $\cos(0.1)$ using T_0, T_2, T_4

Sol ① Maclaurin series : ie. $a = 0$

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x = f(x) \end{aligned} \Rightarrow f^{(k)}(0) = \begin{cases} 1 & \text{if } k = 4m \\ 0 & \text{if } k = 4m+1 \\ -1 & \text{if } k = 4m+2 \\ 0 & \text{if } k = 4m+3 \end{cases}$$

$$\therefore T_\infty(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= \sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} x^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

② $T_{2n+1}(x) = T_{2n}(x)$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}$$

$T_0(x) = 1$

$T_2(x) = 1 - \frac{1}{2}x^2$

$T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

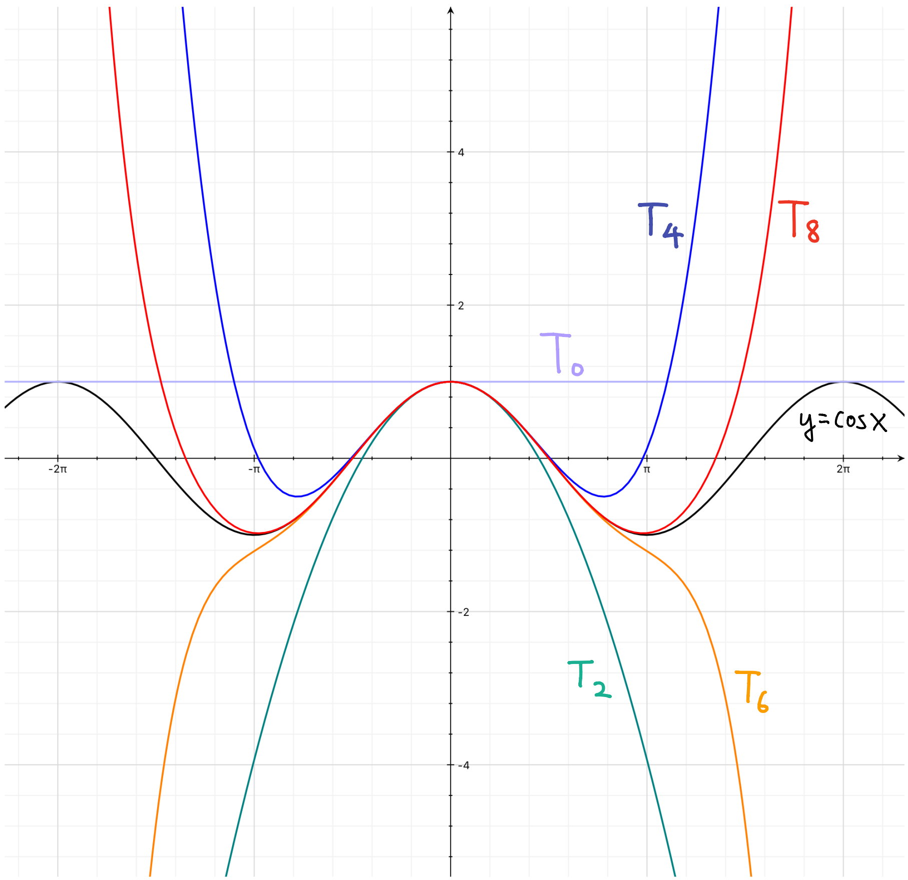
$T_0(0.1) = 1$

$T_2(0.1) = 1 - \frac{1}{2}(0.1)^2 = 0.995$

$T_4(0.1) = \underline{0.9950041}6666\cdots$

Actual value

$\cos(0.1) = \underline{0.9950041}6527$



Graph of $y = \cos x$ and its Taylor Polynomials at 0

Examples of Taylor Series

In the examples below, $T_\infty(x) = f(x)$ for $|x-a| < R$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots \quad \text{for } |x| < 1$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for } x \in \mathbb{R}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for } x \in \mathbb{R}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for } x \in \mathbb{R}$$

$$\circledast \ \ln x = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (x-1)^k = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots \quad \text{for } |x-1| < 1$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots \quad \text{for } |x| < 1$$

Operation on Taylor series

If $T_f(x), T_g(x), T_{f+g}(x)$ is the Taylor series of $f(x), g(x), f(x)+g(x)$ at a respectively

Then $$T_{f+g}(x) = T_f(x) + T_g(x)$$

Similar for other operations

$-$, $\times$, $\div$, composition, differentiation, integration

eg (Addition)

$$\frac{1}{1-x} + \frac{1}{1+x}$$

$$= (1+x+x^2+\dots) + (1-x+x^2-\dots)$$

$$= 2(1+x^2+x^4+\dots)$$

eg (Multiplication)

$$e^x \cdot e^x$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$+ x + x^2 + \frac{x^3}{2} + \dots$$

$$+ \frac{x^2}{2} + \frac{x^3}{2} + \dots$$

$$+ \frac{x^3}{6} + \dots$$

$$= \underbrace{1 + 2x + 2x^2 + \frac{4x^3}{3} + \dots}_{\text{Same as Taylor Series of } e^{2x}}$$

Same as Taylor Series of e^{2x}:

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \dots$$

$$= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \dots$$

eg Find the 3rd order Maclaurin polynomial of ① $e^{2x}\sin x$ ② $e^{\frac{x}{1-x}}$

Sol ① $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

$\therefore$ $e^{2x}\sin x$

$$= \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots\right)\left(x - \frac{x^3}{6} + \cdots\right)$$

$$= x + 2x^2 + 2x^3 - \frac{x^3}{6} + \cdots$$

$$= x + 2x^2 + \frac{11}{6}x^3 + \cdots$$

$\therefore$ $T_3(x) = x + 2x^2 + \frac{11}{6}x^3$

② $\frac{x}{1-x} = x(1 + x + x^2 + \cdots)$

$$= x + x^2 + x^3 + \cdots$$

$\therefore$ $e^{\frac{x}{1-x}}$

$$= 1 + \frac{x}{1-x} + \frac{1}{2}\left(\frac{x}{1-x}\right)^2 + \frac{1}{6}\left(\frac{x}{1-x}\right)^3 + \cdots$$

$$= 1 + (x + x^2 + x^3 + \cdots)$$

$$+ \frac{1}{2}(x + x^2 + x^3 + \cdots)^2$$

$$+ \frac{1}{6}(x + x^2 + x^3 + \cdots)^3 + \cdots$$

$$= 1 + (x + x^2 + x^3 + \cdots) + \frac{1}{2}(x^2 + 2x^3 + \cdots)$$

$$+ \frac{1}{6}(x^3 + \cdots) + \cdots$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + \cdots$$

$\therefore$ $T_3(x) = 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3$

eg Find $\sec x = \dfrac{1}{\cos x}$ (Up to x^4 term)

Method I

Let $\sec x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

Then

$$\begin{aligned}
1 &= (\cos x)(\sec x) \\
&= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right)\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right) \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\
&\qquad - \frac{a_0}{2}x^2 - \frac{a_1}{2}x^3 - \frac{a_2}{2}x^4 + \cdots \\
&\qquad + \frac{a_0}{24}x^4 + \cdots
\end{aligned}$$

Comparing coefficients

$a_0 = 1$

$a_1 = 0$

$a_2 - \frac{a_0}{2} = 0 \quad \Rightarrow a_2 = \frac{1}{2}$

$a_3 - \frac{a_1}{2} = 0 \quad \Rightarrow a_3 = 0$

$a_4 - \frac{a_2}{2} + \frac{a_0}{24} = 0 \Rightarrow a_4 = \frac{5}{24}$

$\therefore \sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots$

Method II

$$\begin{aligned}
&\sec x \\
&= \frac{1}{\cos x} \\
&= \frac{1}{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \cdots\right)} \\
&= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24} + \cdots\right) + \left(\frac{x^2}{2} - \frac{x^4}{24} + \cdots\right)^2 + \cdots \\
&= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24} + \cdots\right) + \left(\frac{x^4}{4} + \cdots\right) \\
&= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots
\end{aligned}$$

same

eg Find Maclaurin Series of $\left(\frac{1}{1-x}\right)^2$

Sol

Method 1 : From definition (Ex)

Method 2 : $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ ⊛

$$\left(\frac{1}{1-x}\right)^2 = (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Method 3 : $\frac{d}{dx}$ ⊛

$$\left(\frac{1}{1-x}\right)^2 = \left(\frac{1}{1-x}\right)'$$

$$= (1 + x + x^2 + x^3 + x^4 + \cdots)'$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1) x^n$$

Ex Show that $\left(\frac{1}{1-x}\right)^3 = \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n$

(Hint : Differentiate $\frac{1}{1-x}$ twice)

eg Find the 6^{th} order Maclaurin polynomial of $f(x) = \arctan x$

Sol Note

$$f'(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

By integration,

$$f(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Put $x = 0 \Rightarrow C = f(0) = \arctan 0 = 0$

$$\therefore T_6(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$$

Find limit using Taylor Series

eg $\displaystyle\lim_{x\to 0} \frac{e^{\sin x} - x - \cos x}{x^2}$

Sol Consider Taylor Series at $a=0$.

$$e^{\sin x} = 1 + \sin x + \frac{(\sin x)^2}{2!} + \cdots$$
$$= 1 + \left(x - \frac{x^3}{3!} + \cdots\right) + \frac{1}{2}\left(x - \frac{x^3}{3!} + \cdots\right)^2 + \cdots$$
$$= 1 + \left(x - \frac{x^3}{3!} + \cdots\right) + \frac{1}{2}\left(x^2 + \cdots\right)$$
$$= 1 + x + \frac{x^2}{2} + \cdots$$

$\therefore e^{\sin x} - x - \cos x$

$$= \left(1 + x + \frac{x^2}{2} + \cdots\right) - x - \left(1 - \frac{x^2}{2!} + \cdots\right)$$
$$= x^2 + \text{terms of degree} \geq 3$$

$$\therefore \frac{e^{\sin x} - x - \cos x}{x^2} = 1 + \text{terms of degree} \geq 1$$

$$\therefore \lim_{x\to 0} \frac{e^{\sin x} - x - \cos x}{x^2} = 1 + 0 = 1$$

Taylor's theorem

How accurate is the approximation $f(x) \approx \underbrace{T_n(x)}_{\text{n-th Taylor Polynomial at } a}$?

Let $f(x) = T_n(x) + \underbrace{R_n(x)}_{\text{Remainder}}$

Thm (Taylor's theorem)

If $x > a$, $f^{(n)}(x)$ exists and is continuous on $[a, x]$

$f^{(n+1)}(x)$ exists on (a, x)

Then $\exists c \in (a, x)$ such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$\therefore f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(x)}$$

Rmk ① Similar statement for $x < a$

② MVT is the special case of Taylor's thm for $n=0$

eg Approximate $\cos(0.1)$ using Maclaurin polynomials $T_n(x)$ such that error $\leq 10^{-6}$

Sol Let $f(x) = \cos x$ $\quad f(x) = T_n(x) + R_n(x)$

By Taylor's theorem, $\exists\, c \in (0, 0.1)$ such that

$$R_n(0.1) = \frac{f^{(n+1)}(c)}{(n+1)!}(0.1 - 0)^{n+1} = \frac{f^{(n+1)}(c)}{10^{n+1}(n+1)!}$$

Note $f^{(n+1)}(c) = \pm \sin c$ or $\pm \cos c$

$$\Rightarrow |f^{(n+1)}(c)| \leq 1$$

$$\Rightarrow \text{error} = |R_n(0.1)| \leq \frac{1}{10^{n+1}(n+1)!} \leq \frac{1}{10^{n+1}}$$

$\therefore$ If $n = 5$, then error $\leq 10^{-6}$

Rmk ①

$$T_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$T_5(0.1) = 0.9950041666\ldots$

$\cos(0.1) = 0.9950041653\ldots$

Rmk ② Similarly, for any $x \in \mathbb{R}$

Taylor's theorem

$$\Rightarrow R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

for some c between 0 and x

$$\Rightarrow \text{error} = |R_n(x)| < \frac{|x|^{n+1}}{(n+1)!}$$

Observation:

i $\displaystyle \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \Rightarrow \lim_{n\to\infty} R_n(x) = 0$

$\displaystyle \therefore \lim_{n\to\infty} T_n(x) = f(x) = \cos x$

ii $\displaystyle \frac{|x|^{n+1}}{(n+1)!} \to 0$ more slowly for large $|x|$

It suggests that

$T_n(x) \to f(x)$ more slowly for large $|x|$.